李昱老師機率數統第 1 次模考 (112 年考生) (滿分 150 分)
機率論: CH1(全), CH2(全), CH3(全)
考試時間: 100 分鐘 [可使用計算機]
[請保留有效位數至少四位以上, 有效位數意指從第一個不是零的位數開始往後計算。]
[計算證明題請將所有過程盡量寫清楚，不完整的內容可能會被扣分。]

1. The foundations of probability are based upon three statements called axioms.
   (a) State the three axioms of probability. (10%)
   (b) Show that the probability of the empty set is zero. (10%)
   (c) For any finite sequence of events $A_1, ..., A_n$ that are mutually exclusive, i.e. $A_i \cap A_j = \phi$ if $i \neq j$, show that $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$. (10%)

[台師數研]

2. From all the possible subsets (including empty subset) of a set with N elements, two subsets A and B are randomly and independently selected. Then find $P(A \subset B)$. (Note that A can equal to B.) (10%)

[清大統研]

3. 某手機公司共有甲、乙兩個生產線，依據統計，甲、乙所製造的手機中分別有 5%, 3% 是瑕疵品。若公司希望在全部的瑕疵品中，由甲生產線所製造的比例不得超過 5/12, 則甲生產線所製造的手機數量可占全部手機產量的百分比至多為何? (10%)

[大學指考]

4. An automobile insurance company has a policy that reimburses a loss up to a benefit limit of 10. This is to say that if the loss is under 10, then the loss is fully paid. If the loss is more than 10, then only 10 is paid. Assume that the loss of a policyholder, X, follows a distribution with the pdf $f(x) = 2x^{-3}$ if $x > 1$; $f(x) = 0$, otherwise. Please calculate the expectation of the benefit paid by the insurance company under the policy described above. (10%)

[中山應數]

5. Let $\overline{X}$ be the mean of a random sample of size $n = 10$ from a distribution with mean $\mu = 70$ and variance $\sigma^2 = 60$. Use Chebyshev's inequality to find a lower bound for $P(65 < \overline{X} < 75)$. (10%)

[清大統研]

6. 企鵝媽媽下蛋, 每批都會固定下十顆蛋, 但是因為各種客觀條件的不同, 使得一批蛋中十顆所共同擁有的孵化率 P 是一個在 (0, 1) 上均勻分配的隨機變數。今企鵝媽媽又下了一批蛋, 請問這批蛋期望會孵出多少隻小企鵝? 孵化隻數的變異數又是多少? (這裡假定每顆蛋如果孵化成功, 只會有一隻小企鵝。) (10%)

7. Let X_1 and X_2 be two i.i.d. random variables with the p.d.f. $f(x) = \frac{e^{-kx}}{k!}, x > 0$.
   (a) What is the value of k if $f(x)$ is a valid probability function? (10%)
   (b) We set $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_1 + X_2}$. What is the joint p.d.f. of Y_1 and Y_2? (10%)

[中興財金]

8. Suppose the joint pdf of X and Y is $f(x,y) = \begin{cases} e^{-y} & , y > x > 0 \\ 0 & , \text{otherwise} \end{cases}$.

(a) Find the marginal pdf of X. (10%)

(b) Find $E(Y|X = x)$. (10%)

(c) Find the correlation coefficient of X and Y. (10%)

[政大統研]

9. 令 $X_1, X_2, X_3 \overset{\text{iid}}{\sim} \mathcal{U}(0,1)$, 則回答以下問題:

(a) 試求 $P(X_1 < X_2)$。(10%)

(b) 試求 $P(X_1 < X_2 < X_3)$。(10%)

[bonus] Let $X_1, X_2, \ldots$ be a sequence of iid $\mathcal{U}(0,1)$ variables and $N = \min\{n \geqslant 2; X_n > X_{n-1}\}$. Find the distribution and the expected value of N. (10%)

[清大統研]

1. The foundations of probability are based upon three statements called axioms.
   (a) State the three axioms of probability. (10%)
   (b) Show that the probability of the empty set is zero. (10%)
   (c) For any finite sequence of events $A_1, \ldots, A_n$ that are mutually exclusive, i.e. $A_i \cap A_j = \phi$ if $i \neq j$, show that $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$. (10%)

[台師數研]

sol: (a) 令 S 為某隨機實驗的樣本空間, $\mathcal{F}$ 為佈於 S 上的一個 σ-域, 函數 $P(\cdot) : \mathcal{F} \longrightarrow \mathbb{R}$ 為一個定義於 $\mathcal{F}$ 上的實值函數, 且滿足以下三點:

(1) **Axiom 1. (**非負性**, non-negativity):**
$P(A) \geqslant 0,\ \forall A \in \mathcal{F}\ \ (\forall A \subset S)$

(2) **Axiom 2. (**歸一性**, normalization):**
$P(S) = 1$

(3) **Axiom 3. (**可數可加性**, countable additivity):**
若 $A_1, A_2, \ldots \in \mathcal{F}\ (A_1, A_2, \ldots \subset S)$, 且 $A_i \cap A_j = \varnothing, \forall i \neq j$, 則

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

我們稱 $P(\cdot)$ 為機率測度 **(probability measure)**, 並且序對 $(S, \mathcal{F}, P)$ 構成一個機率空間 **(probability space)**。

(b) 令 $A_1 = S, A_2 = A_3 = \ldots = \varnothing$

則 $A_i \in \mathcal{F}\ (A_i \subset S), \forall i \in \mathbb{N},\ A_i \cap A_j = \varnothing, \forall i \neq j$, 且 $\bigcup_{i=1}^{\infty} A_i = S \cup \varnothing \cup \varnothing \cup \ldots = S$

由可數可加性 (Axiom 3) 知

$$P(S) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) = P(S) + P(\varnothing) + P(\varnothing) + \ldots$$

$$\Longrightarrow \sum_{i=2}^{\infty} P(\varnothing) = 0 \quad \therefore\ P(\varnothing) = 0$$

(c) 令 $A_k = \varnothing,\ \forall k > n,\ n, k \in \mathbb{N}$ (即 $A_{n+1} = A_{n+2} = \ldots = \varnothing$), 且 $A_i \cap A_j = \varnothing,\ \forall i \neq j$

則 $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{n} A_i \cup \varnothing \cup \varnothing \cup \ldots = \bigcup_{i=1}^{n} A_i$

由可數可加性 (Axiom 3) 與 $P(\varnothing) = 0$ 知

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{n} P(A_i) + \sum_{i=n+1}^{\infty} P(\varnothing) = \sum_{i=1}^{n} P(A_i)$$

$$\therefore\ P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

□

2. From all the possible subsets (including empty subset) of a set with N elements, two subsets A and B are randomly and independently selected. Then find $P(A \subset B)$. (Note that A can equal to B.) (10%)

[清大統研]

sol: 對於一具 N 元素的集合而言, 其可能的子集 (含 $\varnothing$ 與其本身) 共有 2^N 個

若從中任選二個子集, 則可能性共有 $2^N \times 2^N = 4^N$ 種

考慮 B 有 b 個元素, 則滿足條件 (即 $A \subset B$, 其中 A 包含 $\varnothing$ 與 B 本身) 的情況有 2^b 種

故所求為 $P(A \subset B) = \dfrac{\sum_{b=0}^{N} \binom{N}{b} 2^b}{4^N}$

其中由二項式定理可知 $\displaystyle\sum_{b=0}^{N} \binom{N}{b} 2^b = \sum_{b=0}^{N} \left[\binom{N}{b} 2^b \times 1^{N-b} \right] = (2+1)^N = 3^N$

故所求為 $P(A \subset B) = \dfrac{\sum_{b=0}^{N} \binom{N}{b} 2^b}{4^N} = \dfrac{3^N}{4^N} = \left(\dfrac{3}{4}\right)^N$

□

3. 某手機公司共有甲、乙兩個生產線，依據統計，甲、乙所製造的手機中分別有 5%, 3% 是瑕疵品。若公司希望在全部的瑕疵品中，由甲生產線所製造的比例不得超過 5/12, 則甲生產線所製造的手機數量可占全部手機產量的百分比至多為何? (10%)

[大學指考]

sol: 假設甲生產線所製造的手機數量可占全部手機產量的百分比為 p

則所有瑕疵品的比例為 $p \times 0.05 + (1-p) \times 0.03$

在全部的瑕疵品中, 由甲生產線所製造的比例為 $\dfrac{p \times 0.05}{p \times 0.05 + (1-p) \times 0.03}$

依題意可令 $\dfrac{p \times 0.05}{p \times 0.05 + (1-p) \times 0.03} \leqslant \dfrac{5}{12}$

$\therefore\ p \leqslant 0.3$, 即甲生產線所製造的手機數量至多可佔全部手機產量的 30%。

□

4. An automobile insurance company has a policy that reimburses a loss up to a benefit limit of 10. This is to say that if the loss is under 10, then the loss is fully paid. If the loss is more than 10, then only 10 is paid. Assume that the loss of a policyholder, X, follows a distribution with the pdf $f(x) = 2x^{-3}$ if $x > 1$; $f(x) = 0$, otherwise. Please calculate the expectation of the benefit paid by the insurance company under the policy described above. (10%)

[中山應數]

sol: 依題意, 可令 $Y = \begin{cases} X & , \text{if } X < 10 \\ 10 & , \text{if } X \geqslant 10 \end{cases}$ 表示所獲得之支付額, 則所求為

$$E(Y) = \int_1^{10} x \cdot 2x^{-3}\, dx + 10 \times \int_{10}^{\infty} 2x^{-3}\, dx = \left[\frac{2}{-1} x^{-1} \right]_1^{10} + 10 \times \left[\frac{2}{-2} x^{-2} \right]_{10}^{\infty}$$

$$= \frac{-2}{10} + 2 + 10 \times \left(0 + \frac{1}{100} \right) = \frac{19}{10} = 1.9$$

□

5. Let $\overline{X}$ be the mean of a random sample of size $n = 10$ from a distribution with mean $\mu = 70$ and variance $\sigma^2 = 60$. Use Chebyshev's inequality to find a lower bound for $P(65 < \overline{X} < 75)$. (10%)

[清大統研]

sol: 由題意可知 $X_1, ..., X_{10} \overset{\text{iid}}{\sim} (\mu = 70, \sigma^2 = 60)$, 則可知 $E(\overline{X}) = 70$, $\mathrm{Var}(\overline{X}) = \dfrac{60}{10} = 6$

則由柴比雪夫不等式 (Chebyshev's inequality) 可知

$$P(65 < \overline{X} < 75) = P\Big(|\overline{X} - 70| < 5\Big) \geqslant 1 - \frac{6}{5^2} = \frac{19}{25}$$

□

6. 企鵝媽媽下蛋, 每批都會固定下十顆蛋, 但是因為各種客觀條件的不同, 使得一批蛋中十顆所共同擁有的孵化率 P 是一個在 (0, 1) 上均勻分配的隨機變數。今企鵝媽媽又下了一批蛋, 請問這批蛋期望會孵出多少隻小企鵝? 孵化隻數的變異數又是多少? (這裡假定每顆蛋如果孵化成功, 只會有一隻小企鵝。) (10%)

sol: 由題目知道, 孵化率 $P \sim \mathcal{U}(0, 1)$, 則令 X 表示一批 10 顆蛋中孵化的個數

依照題意可知 $X|P = p \sim \mathrm{Bin}(10, p)$, 由雙重期望值定理可知, 所求期望值為

$$E\big[E(X|P)\big] = E(10 \times P) = 10 \times E(P) = 5$$

又由變異數分解定理可知, 所求變異數為

$$\mathrm{Var}(X) = E\big[\mathrm{Var}(X|P)\big] + \mathrm{Var}\big[E(X|P)\big] = E\big[10p(1-p)\big] + \mathrm{Var}(10p)$$

$$= 10E(p) - 10E(p^2) + 10^2\mathrm{Var}(p) = 10 \times \frac{0+1}{2} - 10 \times \frac{0^2 + 0 \times 1 + 1^2}{3} + 10 \times \frac{(1-0)^2}{12}$$

$$= \frac{10}{2} - \frac{10}{3} + \frac{10}{12} = 2.5$$

□

7. Let X_1 and X_2 be two i.i.d. random variables with the p.d.f. $f(x) = \dfrac{e^{-kx}}{k!}, x > 0$.

   (a) What is the value of k if $f(x)$ is a valid probability function? (10%)

   (b) We set $Y_1 = X_1 + X_2$ and $Y_2 = \dfrac{X_1}{X_1 + X_2}$. What is the joint p.d.f. of Y_1 and Y_2? (10%)

[中興財金]

sol: (a) 由 pdf 的性質檢查可知

$$1 = \int_0^\infty \frac{e^{-kx}}{k!}dx = \left[\frac{e^{-kx}}{(-k)\cdot k!}\right]_0^\infty = -\frac{1}{(-k)\cdot k!} \quad \therefore \quad k = 1$$

(b) 由 $Y_1 = X_1 + X_2$, $Y_2 = \dfrac{X_1}{X_1 + X_2} \implies X_1 = Y_1Y_2$, $X_2 = Y_1(1 - Y_2)$

$$\implies \mathcal{J} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{\partial(y_1y_2)}{\partial y_1} & \frac{\partial(y_1y_2)}{\partial y_2} \\ \frac{\partial y_1(1-y_2)}{\partial y_1} & \frac{\partial y_1(1-y_2)}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 1-y_2 & -y_1 \end{vmatrix} = -y_1$$

$$\implies f_{Y_1Y_2}(y_1, y_2) = f_{X_1X_2}\big(y_1y_2, y_1(1-y_2)\big)|\mathcal{J}|$$

$$= e^{-y_1}|-y_1| = y_1e^{-y_1},\ y_1 > 0, 0 < y_2 < 1$$

□

8. Suppose the joint pdf of X and Y is $f(x,y)=\begin{cases} e^{-y} & , y>x>0 \\ 0 & , \text{otherwise} \end{cases}$.

(a) Find the marginal pdf of X. (10%)

(b) Find $E(Y|X=x)$. (10%)

(c) Find the correlation coefficient of X and Y. (10%)

[政大統研]

sol: (a) $f_X(x)=\int_x^{\infty} e^{-y}\,dy=\left[-e^{-y}\right]_x^{\infty}=0-\left(-e^{-x}\right)=e^{-x},\ x>0$

此即說明 $X\sim \text{Exp}(\beta=1)$

(b) $f_{Y|X}(y|x)=\dfrac{f_{XY}(x,y)}{f_X(x)}=\dfrac{e^{-y}}{e^{-x}}=e^{-(y-x)},\ y>x>0$

$\Longrightarrow\ Y|X=x\sim \text{Exp}(\beta=1,\theta=x)$, 故可知 $E(Y|X=x)=\theta+\beta=x+1,\ x>0$

(c) 由前述結果可知 $\text{Var}(X)=1^2=1$, 且由變異數分解定理可知

$$\text{Var}(Y)=E\big[\text{Var}(Y|X)\big]+\text{Var}\big[E(Y|X)\big]$$
$$=E(1^2)+\text{Var}(X+1)=1+\text{Var}(X)=1+1=2$$

又由母體線性迴歸方程式可知, $E(Y|X=x)=1\times x+1$

$$\Longrightarrow\ 1=\rho_{XY}\frac{\sigma_Y}{\sigma_X}=\rho_{XY}\frac{\sqrt{2}}{\sqrt{1}}\quad \therefore\ \rho_{XY}=\frac{1}{\sqrt{2}}$$

□

9. 令 $X_1,X_2,X_3\overset{\text{iid}}{\sim}\mathcal{U}(0,1)$, 則回答以下問題:

(a) 試求 $P(X_1<X_2)$。(10%)

(b) 試求 $P(X_1<X_2<X_3)$。(10%)

sol: (a) [法一]

由題意可知 $f_{X_1X_2}(x_1,x_2)=f_{X_1}(x_1)\,f_{X_2}(x_2)=1,\ 0<x_1,x_2<1$, 則

$$P(X_1<X_2)=\int_0^1\int_0^{x_2} f_{X_1X_2}(x_1,x_2)\,dx_1\,dx_2=\int_0^1\int_0^{x_2} 1\,dx_1\,dx_2=\left[\frac{1}{2}x_2^2\right]_0^1=\frac{1}{2}$$

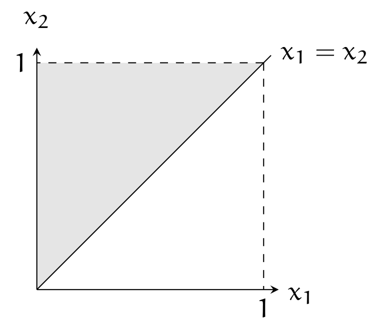
[法二]

本題所求範圍如右圖所示

又由聯合分配為均勻分配可知

機率即為所求範圍在值域中之佔比

即 $P(X_1<X_2)=1\times 1\times\dfrac{1}{2}=\dfrac{1}{2}$



(b) [法一]

由題意可知 $f_{X_1X_2X_3}(x_1,x_2,x_3)=f_{X_1}(x_1)\,f_{X_2}(x_2)\,f_{X_3}(x_3)=1,\ 0<x_1,x_2,x_3<1$, 則

$$P(X_1 < X_2 < X_3) = \int_0^1 \int_0^{x_3} \int_0^{x_2} f_{X_1X_2X_3}(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3$$

$$= \int_0^1 \int_0^{x_3} \int_0^{x_2} 1\, dx_1\, dx_2\, dx_3 = \left[\frac{1}{6}x_3^3\right]_0^1 = \frac{1}{6}$$

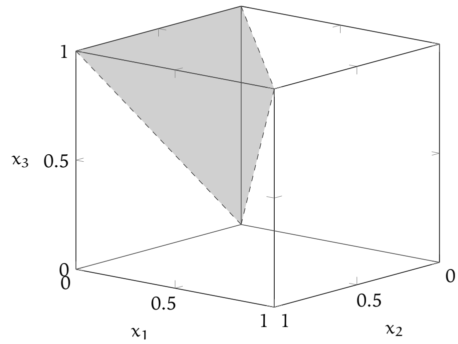
[法二]

本題所求範圍如右圖所示

又由聯合分配為均勻分配可知

機率即為所求範圍在值域中之佔比，即

$$P(X_1 < X_2 < X_3) = 1 \times 1 \times \frac{1}{2} \times 1 \times \frac{1}{3} = \frac{1}{6}$$



□

[bonus] Let $X_1, X_2, \ldots$ be a sequence of iid $\mathcal{U}(0,1)$ variables and $N = \min\{n \geqslant 2; X_n > X_{n-1}\}$. Find the distribution and the expected value of N. (10%)

[清大統研]

<u>**sol:**</u> 依題意可知，$N = n$ 表示首次使得該次的樣本比前一次樣本大的情況，發生在第 n 次

亦即，直到第 n 個樣本為止，前面的 $n-1$ 個樣本為 $X_1 > X_2 > \ldots > X_{n-1}$，故我們有

$$P(N = n) = P(X_1 > X_2 > \ldots > X_{n-1}) - P(X_1 > X_2 > \ldots > X_{n-1} > X_n)$$

$$= \frac{1}{(n-1)!} - \frac{1}{n!} = \frac{1}{(n-1)!} \cdot \left[1 - \frac{1}{n}\right] = \frac{1}{n} \cdot \frac{1}{(n-2)!},\ n = 2, 3, \ldots.$$

$$\Longrightarrow E(N) = \sum_{n=2}^{\infty} n \times \frac{1}{n} \cdot \frac{1}{(n-2)!} = \sum_{n=2}^{\infty} \frac{1}{(n-2)!} \qquad (\text{令 } m = n-2)$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} = e^1 = e$$

<u>筆記</u> 事實上，包含第九題，不論是不是 $\mathcal{U}(0,1)$ 分配，只要是一組連續變數的隨機樣本，由於獨立同分配的緣故，其任意的一組順序 (譬如 $X_1 > X_2 > X_3$ 或是 $X_2 > X_1 > X_3$ 等等) 發生的機率都是 $1/n!$，由此可以快速地計算出有序樣本的機率，搭配題目定義的各種隨機變數 (譬如此處的 N) 所代表的等價事件，可以求出它們的機率分配，剩下的事情只是考驗大家的基礎數學計算能力而已。